

Positivity in equivariant Schubert calculus

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1 Introduction

Let $X = G/B$ be the flag variety of a complex semisimple group G with $B \supset T$ a Borel subgroup and maximal torus, respectively. The homology $H_*(X)$ has as a basis the fundamental classes $[X_w]$ of Schubert varieties $X_w \subset X$; if $\{x_w\} \subset H^*(X)$ is the corresponding dual basis for cohomology, the cup product, expressed in this basis, has nonnegative coefficients:

$$x_u x_v = \sum a_{uv}^w x_w \quad (1.1)$$

where a_{uv}^w are nonnegative integers.

The T -equivariant cohomology and Chow groups of the flag variety have been described by [A], [KK], [Br]. One reason to study these groups is that they provide a way to compute the coefficients in the multiplication in ordinary cohomology. In addition, the equivariant groups are related to degeneracy loci in algebraic geometry (see [F2], [F3], [P-R], [G]), which in turn are related to the double Schubert polynomials first defined in combinatorics by [L-S].

Peterson [P] recently conjectured that the equivariant cohomology groups of the flag variety have a positivity property generalizing (1.1). The T -equivariant cohomology $H_T^*(X)$ is a free module over $H_T^*(pt)$ with a basis dual (in a suitable sense; see Section 2) to the equivariant fundamental classes $[X_w]_T$; again we call this basis $\{x_w\}$. Now $H_T^*(pt)$ is isomorphic to the polynomial ring $S(\hat{T}) = \mathbb{Z}[\lambda_1, \dots, \lambda_n]$, where $\lambda_1, \dots, \lambda_n$ is a basis for the free abelian group \hat{T} of characters of T . Let $\alpha_1, \dots, \alpha_n$ denote the simple roots in \hat{T} (chosen so that the roots of $\mathfrak{b} = \text{Lie } B$ are positive). In the equivariant setting, we can again expand the product $x_u x_v$ in the form (1.1), but now the a_{uv}^w are in $H_T^*(pt)$ – in other words, they are polynomials. Peterson's conjecture is that when each a_{uv}^w is written as a sum of monomials in the α_i , the coefficients are all nonnegative. In this paper we prove the conjecture, not just for finite-dimensional flag varieties, but in the general Kač-Moody setting. An immediate corollary is a conjecture of Billey [Bi].

The methods of this paper are those used by Kumar and Nori [KN]. In that paper, the authors prove the nonnegativity result (1.1) in ordinary cohomology for the flag variety of a Kač-Moody group. As they observe, the difficulty in proving this result is that in the Kač-Moody case, unlike the finite dimensional case, the flag variety is not in general

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a homogeneous space. However, it is approximated by finite-dimensional varieties, each of which has an action of a unipotent group with finitely many orbits. The main result of [KN] is that for such varieties, the cup product has nonnegative coefficients (with respect to a suitable basis); the result for the flag variety follows. A similar problem arises in equivariant cohomology. The equivariant cohomology of X is by definition the cohomology of a “mixed space” X_T , which, although infinite-dimensional, can be approximated by finite-dimensional varieties. As in the situation considered by Kumar and Nori, the space X_T is not a homogeneous space. But unlike their situation, the finite-dimensional approximations to X_T do not (as far as I know) have actions of unipotent groups with finitely many orbits, so we cannot apply their result. Instead, by adapting their proof to the equivariant setting, and using a relation in equivariant cohomology (or Chow groups) observed by Brion, we are able to deduce an equivariant analogue of the main result of [KN]. The equivariant nonnegativity result for the flag variety follows immediately.

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2 Preliminaries

We will work with schemes over the ground field \mathbb{C} and assume (to freely apply the results of [F, Ch.19]) that all schemes considered admit closed embeddings into nonsingular schemes. We use equivariant cohomology and Borel-Moore homology with integer coefficients as our main tools; H_*X will denote the Borel-Moore homology of X . For smooth varieties, we could alternatively use equivariant Chow groups, but for nonsmooth varieties, the Chow “cohomology” theory is not as well understood and for this reason we use (equivariant) cohomology and Borel-Moore homology groups. In this section we recall some basic facts about these groups; for more background, see [Br2] or [E-G]. We also prove, for lack of a reference, equivariant versions of several familiar non-equivariant results.

Let X be a scheme with an action of a linear algebraic group G . Let V be a representation of G and U an open subset of V such that G acts freely on U and such that the (complex) codimension of $V - U$ in V is greater than $\dim X - i/2$. View G as acting on the right on U , and on the left on X ; then G acts on $U \times X$ by $g \cdot (u, x) = (ug^{-1}, gx)$.¹ Define $U \times^G X$ to be $(U \times X)/G$. The equivariant cohomology and Borel-Moore homology of X are, by definition,

$$\begin{aligned} H_G^i(X) &= H^i(U \times^G X) \\ H_i^G(X) &= H_{i+2(\dim V - \dim G)}(U \times^G X). \end{aligned}$$

These groups are independent of the choice of V and U provided the codimension condition is satisfied. For this reason we often denote $U \times^G X$ by X_G (omitting U from the notation). The quotient U/G is a finite-dimensional approximation to the classifying space BG introduced in Chow theory by Totaro [T]. We will frequently write BG when we mean such a finite-dimensional approximation.

The equivariant cohomology of a point we denote by H_G^* . Both $H_G^*(X)$ and $H_*^G(X)$ are modules for H_G^* . $H_G^*(X)$ has a natural ring structure, and $H_*^G(X)$ is a module for this

¹Alternatively, we could let G act on the left on U and then take the diagonal action on $U \times X$.

ring. Any G -stable closed subvariety $Y \subset X$ has a fundamental class $[Y]_G$ in $H_{2\dim Y}^G(X)$. There is a natural map $\cap[X]_G : H_G^*(X) \rightarrow H_*^G(X)$; if X is smooth this is an isomorphism. In particular, we will always identify $H_*^G(pt)$ with H_G^* .

Let $\pi^X : X \rightarrow pt$ denote the projection. If X is proper, this induces an H_G^* -linear map $\pi_*^X : H_*^G(X) \rightarrow H_*^G(pt) \cong H_G^*$. In this case, there is a pairing $(,) : H_G^*(X) \otimes H_*^G(X) \rightarrow H_G^*$ taking $x \otimes C$ to $\pi_*^X(x \cap C)$. We will sometimes write this pairing as $\int_C x$, and if $C = [Y]_G$, we will abuse notation and write it as $\int_Y x$. The pairing has the property that given $f : X_1 \rightarrow X_2$, we have

$$(f^*x_2, C_1) = (x_2, f_*C_1). \quad (2.2)$$

(Proof: $(f^*x_2, C_1) = \pi_*^{X_1}(f^*x_2 \cap C_1) = \pi_*^{X_2}f_*(f^*x_2 \cap C_1) = \pi_*^{X_2}(x_2 \cap f_*C_1) = (x_2, f_*C_1)$.)

The map $X \times^G U \rightarrow U/G$ is a fibration with fiber X , and pullback to a fiber yields a map $H_G^*(X) \rightarrow H^*(X)$. There is also a Gysin morphism $H_*^G(X) \rightarrow H_*(X)$.

A variety X is said to be paved by affines if it can be written as a finite disjoint union $X = \coprod X_i^0$ where X_i^0 is isomorphic to affine space \mathbb{A}^{d_i} for some d_i . As is well known (see e.g. [KN]) the Borel-Moore homology $H_*(X)$ is the free \mathbb{Z} -module generated by the fundamental classes $[X_i]$ (where X_i is the closure of X_i^0); the odd-dimensional Borel-Moore homology vanishes.

Part (b) of the next proposition and the remark following are from [A] (Prop. 2.5.1 and 2.4.1), with a somewhat different proof.

Proposition 2.1 *Suppose the G -variety X has a pairing by G -invariant affines X_i^0 . Then*

- (a) $H_*^G(X)$ is a free H_G^* -module with basis $\{[X_i]_G\}$.
- (b) *Suppose in addition that X is complete and that H_G^* is torsion-free. Then there exist classes x_i (of degree $\dim X_i$) in $H_*^G(X)$ which form a basis for $H_*^G(X)$ as H_G^* -module, such that the bases $\{[X_i]_G\}$ and $\{x_i\}$ are dual in the sense that $\int_{X_i} x_j = \delta_{ij}$.*

Proof: (a) Let X_k^0 be open in X , and $Y = X - X_k^0$; then there is a long exact sequence of H_G^* -modules

$$\rightarrow H_{i+1}^G(X_k^0) \rightarrow H_i^G(Y) \rightarrow H_i^G(X) \rightarrow H_i^G(X_k^0) \rightarrow \dots$$

Since X_k^0 is isomorphic to affine space, $H_*^G(X_k^0)$ is a free H_G^* -module of rank 1, generated by $[X_k^0]_G$. Hence all the odd equivariant homology of X_k^0 vanishes, by induction the same holds for Y , and then by the long exact sequence it holds for X . Thus we have a short exact sequence of H_G^* -modules:

$$0 \rightarrow H_*^G(Y) \rightarrow H_*^G(X) \rightarrow H_*^G(X_k^0) \rightarrow 0.$$

This is split by the H_G^* -linear map $H_*^G(X_k^0) \rightarrow H_*^G(X)$ taking $[X_k^0]_G$ to $[X_k]_G$. Induction implies (a).

(b) Because the odd ordinary cohomology of X vanishes, the pullback to a fiber $H_G^*(X) \rightarrow H^*(X)$ is surjective (this is because the spectral sequence of the fibration $X_G \rightarrow B_G$ degenerates at E_2). If $\{y_i\}$ are any classes of pure degree in $H_G^*(X)$ which pull back to a basis of $H^*(X)$ (we may assume $\deg y_i = \dim X_i$), then by the Leray-Hirsch theorem [Sp], $H_G^*(X)$ is a free H_G^* -module with basis $\{y_i\}$. Claim: The matrix $A = (a_{ij})$ with

entries $a_{ij} \in H_G^*$ defined by $a_{ij} = \int_{X_i} y_j$ is invertible. This can be seen by slightly modifying the arguments of [G, Theorem 4.1]. For, we may assume that the X_i are numbered so that the dimension increases as i increases. Now, $\int_{X_i} y_j = 0$ unless $\deg y_j \geq \dim X_i$. This implies that the matrix (a_{ij}) is block upper triangular (here a block of the matrix corresponds to the set of (i, j) with $\dim X_i = d$, $\dim X_j = e$, for fixed d and e). Moreover, the diagonal blocks are invertible matrices of scalars (as, for any fixed d , the entries in the corresponding diagonal block are just the values $([X_i], y'_j)$, where y'_j is the pullback to a fiber of y_j ; $\{[X_i]\}$ is a basis for $H_{2d}(X)$ and $\{y'_j\}$ a basis for $H^{2d}(X)$). Hence the matrix A is invertible, as claimed.

Let $B = A^{-1} = (b_{ij})$ and define $x_j = \sum_i b_{ij} y_i$. Then $\{x_j\}$ is a basis of $H_G^*(X)$ dual to $\{[X_i]_G\}$. Indeed,

$$\int_{X_i} x_j = \sum_k \int_{X_i} b_{kj} y_k = \sum_k b_{kj} \int_{X_i} y_k = \sum_k b_{kj} a_{ik} = \delta_{ij}. \quad (2.3)$$

Note that the dual basis is uniquely determined by (2.3), as can be seen by expressing one dual basis in terms of another. Because the $[X_i]_G$ have pure degree $\dim X_i$, the elements x_j of the dual basis must have degree $\dim X_j$. For, if Y is an irreducible closed subvariety of X , and $y \in H_G^k(X)$, then $\int_Y y$ has degree $k - \dim Y$. Hence if we replace each x_j by its component in degree $\dim X_j$, we still have a dual basis. As the dual basis is unique, each x_j must have degree $\dim X_j$. \square

Remarks. (1) The conditions $\int_{X_i} x_j = \delta_{ij}$ imply that under the map $H_G^*(X) \rightarrow H^*(X)$, the images x'_i of x_i form a basis of $H^*(X)$ dual to the basis $\{[X_i]\}$ of $H_*(X)$.

(2) This result and the next are also valid with coefficients in a field; then H_G^* is automatically torsion-free.

For a variety X paved by G -invariant affines as above, we have the following description of the product on $H_G^*(X)$ in terms of the diagonal morphism. The non-equivariant version of this result was used by [KN]. The equivariant version was mentioned in [P] for the flag variety; the general proof is the same. Note that the diagonal morphism $\delta : X \rightarrow X \times X$ is G -equivariant (G acting diagonally on $X \times X$).

Proposition 2.2 *Let X be a G -variety with a paving by G -invariant affines X_i^0 ; assume H_G^* is torsion-free. Let X_i and x_i be as in the previous proposition. We can write $\delta_*[X_k]_G = \sum_{i,j} a_{ij} [X_i \times X_j]_G$, where $a_{ij} \in H_G^*$. The product in $H_G^*(X)$ is given by*

$$x_i x_j = \sum_k a_{ij}^k x_k.$$

Proof: We can write $\delta_*[X_k]_G$ in the form claimed because the classes $[X_i \times X_j]_G$ form a basis for $H_G^*(X \times X)$ as H_G^* -module.

Let $q_i : X \times X \rightarrow X$ denote the i -th projection. As in the non-equivariant case, the product on $H_G^*(X)$ is given by

$$c_1 \cdot c_2 = \delta^*(q_1^* c_1 \cdot q_2^* c_2)$$

for $c_1, c_2 \in H_G^*(X \times X)$. (This can be seen by considering the composition

$$X_G \xrightarrow{\delta_G} (X \times X)_G \cong X_G \times_{BG} X_G \xrightarrow{i} X_G \times X_G$$

and noting that the product on $H^*(X_G)$ is given by

$$\zeta_1 \cdot \zeta_2 = (i \circ \delta_G)^*(pr_1^* \zeta_1 \cdot pr_2^* \zeta_2)$$

where $pr_i : X_G \times X_G \rightarrow X_G$ is the projection and $\zeta_i \in H^*(X_G)$. Choosing ζ_i to represent $c_i \in H_G^*(X)$, the assertion follows easily.)

The preceding proposition shows that if X is paved by invariant affines, then $H_G^*(X)$ and $H_*^G(X)$ are free H_G^* -modules, with a perfect pairing

$$(\ , \) : H_G^*(X) \otimes_{H_G^*} H_*^G(X) \rightarrow H_G^*.$$

Using this, we can identify

$$H_G^*(X) = \text{Hom}_{H_G^*}(H_*^G(X), H_G^*).$$

Therefore, to show that $x_i x_j = \sum_k a_{ij}^k x_k$, it is enough to show that for all $\nu \in H_*^G(X)$, we have

$$(x_i x_j, \nu) = \left(\sum_k a_{ij}^k x_k, \nu \right) = \sum_k a_{ij}^k (x_k, \nu).$$

In fact, it is enough to check this when ν is one of the basis elements $[X_k]_G$, i.e., it is enough to show

$$(x_i x_j, [X_k]_G) = a_{ij}^k.$$

Now

$$\begin{aligned} (x_i x_j, [X_k]_G) &= (\delta^*(q_1^* x_i \cdot q_2^* x_j), [X_k]_G) \\ &= (q_1^* x_i \cdot q_2^* x_j, \delta_* [X_k]_G) \\ &= \sum_{m,n} a_k^{mn} (q_1^* x_i \cdot q_2^* x_j, [X_m \times X_n]_G). \end{aligned}$$

By definition of the pairing, $(q_1^* x_i \cdot q_2^* x_j, [X_m \times X_n]_G) = \pi_*^{X \times X} (q_1^* x_i \cdot q_2^* x_j \cap [X_m \times X_n]_G)$. This is computed using the fibrations $X_G \rightarrow B_G$ and $(X \times X)_G = X_G \times_{B_G} X_G \xrightarrow{\pi_G} B_G$. By the next lemma, the result is equal to

$$\pi_*^X (x_i \cap [X_m]_G) \cdot \pi_*^X (x_j \cap [X_n]_G)$$

which is 1 if $i = m$ and $j = n$, and 0 otherwise. We conclude $(x_i x_j, [X_k]_G) = a_k^{ij}$, as desired. \square

Lemma 2.3 *Let $\rho_i : X_i \rightarrow Y$ ($i = 1, 2$) be fibrations with ρ_i proper, $\pi : X_1 \times_Y X_2 \rightarrow Y$, $q_i : X_1 \times_Y X_2 \rightarrow X_i$ the projections. Let $Z_i \subset X_i$ be closed subvarieties and $\alpha_i \in H^*(X_i)$. Assume Y is smooth, and identify $H_*(Y)$ with $H^*(Y)$. Then*

$$\pi_*(q_1^* \alpha_1 \cdot q_2^* \alpha_2 \cap [Z_1 \times_Y Z_2]) = \rho_{1*}(\alpha_1 \cap [Z_1]) \cdot \rho_{2*}(\alpha_2 \cap [Z_2])$$

where on the right hand side the product is taken in $H^*(Y)$.

Proof: We have a Cartesian diagram

$$\begin{array}{ccc} X_1 \times_Y X_2 & \xrightarrow{\Delta} & X_1 \times X_2 \\ \downarrow \pi & & \downarrow \Pi \\ Y & \xrightarrow{\delta} & Y \times Y \end{array}$$

Because Y is smooth, δ (and hence Δ) are regular embeddings, so there are Gysin maps δ^* and Δ^* on homology. Claim: In $H_*(X_1 \times_Y X_2)$,

$$q_1^* \alpha_1 \cdot q_2^* \alpha_2 \cap [Z_1 \times_Y Z_2] = \Delta^*((\alpha_1 \cap [Z_1]) \times (\alpha_2 \cap [Z_2])).$$

To prove this, first note that (with $pr_i : X_1 \times X_2 \rightarrow X_i$ denoting the projection) $q_1^* \alpha_1 \cdot q_2^* \alpha_2 = \Delta^*(pr_1^* \alpha_1 \cdot pr_2^* \alpha_2) = \Delta^*(\alpha_1 \times \alpha_2)$ (cf. [Mu, p. 351]). Next, $[Z_1 \times_Y Z_2] = \Delta^*[Z_1 \times Z_2]$, since $Z_1 \times Z_2$ and $\Delta(X \times_Y X)$ are subvarieties of $X_1 \times X_2$ whose intersection at smooth points is transverse. Hence (noting that $[Z_1 \times Z_2] = [Z_1] \times [Z_2]$ by [F, p. 377])

$$\begin{aligned} q_1^* \alpha_1 \cdot q_2^* \alpha_2 \cap [Z_1 \times_Y Z_2] &= \Delta^*(\alpha_1 \times \alpha_2) \cap \Delta^*[Z_1 \times Z_2] \\ &= \Delta^*((\alpha_1 \times \alpha_2) \cap ([Z_1] \times [Z_2])) \\ &= \Delta^*((\alpha_1 \cap [Z_1]) \times (\alpha_2 \cap [Z_2])) \end{aligned}$$

proving the claim.

To complete the proof of the lemma, we compute:

$$\begin{aligned} \pi_*(q_1^* \alpha_1 \cdot q_2^* \alpha_2 \cap [Z_1 \times_Y Z_2]) &= \pi_* \Delta^*((\alpha_1 \cap [Z_1]) \times (\alpha_2 \cap [Z_2])) \\ &= \delta^* \Pi_*((\alpha_1 \cap [Z_1]) \times (\alpha_2 \cap [Z_2])) \\ &= \delta^*(\rho_{1*}(\alpha_1 \cap [Z_1]) \times \rho_{2*}(\alpha_2 \cap [Z_2])) \\ &= \rho_{1*}(\alpha_1 \cap [Z_1]) \cdot \rho_{2*}(\alpha_2 \cap [Z_2]). \end{aligned}$$

This proves the lemma. \square

3 The positivity theorem

In this section we prove the positivity result about multiplication in equivariant cohomology (Theorem 3.1). As in the non-equivariant case considered by Kumar and Nori, it is deduced from a result about invariant cycles (Theorem 3.2). In the non-equivariant setting, Hirschowitz [Hi] proved that for a projective scheme with an action of a connected solvable group B , any effective cycle is rationally equivalent to a B -invariant effective cycle. Kumar and Nori gave a different proof of this result (without assuming projectivity) in the special case of unipotent groups, and the proof of Theorem 3.2 is adapted from their proof.

In this section, T will denote an algebraic torus (i.e. product of multiplicative groups \mathbb{G}_m) with Lie algebra $\mathfrak{t} = \text{Lie } T$, and $\hat{T} \subset \mathfrak{t}^*$ the group of characters of T . The equivariant cohomology group H_T^* can be identified with the polynomial ring $S(\hat{T})$, the symmetric algebra on the free abelian group \hat{T} .

Theorem 3.1 *Let B be a connected solvable group with unipotent radical N and Levi decomposition $B = TN$. Let $\alpha_1, \dots, \alpha_d \in \hat{T}$ denote the weights of T on $\mathfrak{n} = \text{Lie } N$. Let X*

be a complete B -variety on which N acts with finitely many orbits X_1^0, \dots, X_n^0 . These are a paving of X by B -stable affines; let X_1, \dots, X_n denote the closures, so $\{[X_1]_T, \dots, [X_n]_T\}$ are a basis for $H_*^T(X)$. Let $\{x_1, \dots, x_n\}$ denote the dual basis of $H_T^*(X)$. Write

$$x_i x_j = \sum_k a_{ij}^k x_k$$

with $a_{ij}^k \in H_T^* = S(\hat{T})$. Then each a_{ij}^k can be written as a sum of monomials $\alpha_1^{i_1} \cdots \alpha_d^{i_d}$, with nonnegative integer coefficients.

Note that the constant term in each a_{ij}^k (i.e., the coefficient of $\alpha_1^0 \cdots \alpha_d^0$) is nonnegative by the above theorem. This is the coefficient that occurs in the multiplication in the ordinary cohomology $H^*(X)$. The reason is that our hypotheses imply $H^*(X) = H_T^*(X)/H_T^{>0} \cdot H_T^*(X)$ (see [GKM]).

The next result is the key ingredient in the proof of Theorem 3.1. In this theorem, N is not assumed to act with finitely many orbits. The result also holds with equivariant Chow groups in place of equivariant Borel-Moore homology.

Theorem 3.2 *Let B be a connected solvable group with unipotent radical N , and let $T \subset B$ be a maximal torus, so $B = TN$. Let $\alpha_1, \dots, \alpha_d \in \hat{T}$ denote the weights of T acting on $\mathfrak{n} = \text{Lie } N$. Let X be a scheme with a B -action and Y a T -stable subvariety of X . Then there exist B -stable subvarieties D_1, \dots, D_r of X such that in $H_*^T(X)$,*

$$[Y]_T = \sum f_i [D_i]_T$$

where each $f_i \in H_T^*$ can be written as a linear combination of monomials in $\alpha_1, \dots, \alpha_d$ with nonnegative integer coefficients.

The following lemma was pointed out to me by Michel Brion.

Lemma 3.3 *Suppose the connected solvable group $B = TN$ acts on X and that N has finitely many orbits on X . Then each N -orbit is B -stable (in fact, the B -orbit of a T -fixed point).*

Proof: B has finitely many orbits on X (as the subgroup N does); as each N -orbit is N -stable, it is a finite union of N -orbits. Let $B \cdot x' \simeq B/B'$ be an orbit, where B' is the stabilizer of x' . As each N -orbit is isomorphic to affine space (see e.g. [KN]), the odd cohomology of $B \cdot x'$ vanishes, so B' must contain a maximal torus of B . As all maximal tori of B are conjugate [Bo, Corollary 11.3], there is some $b \in B$ such that $B' = bB_1b^{-1}$, where $B_1 \supset T$. Then $B \cdot x' = B \cdot x$ where $x = b^{-1}x'$; moreover B_1 is the stabilizer of x . Hence $B \cdot x$ is the N -orbit of the T -fixed point x . \square

Proof of Theorem 3.1: The group $\tilde{B} = T \cdot (N \times N)$ (semi-direct product) acts on $X \times X$ by $t \cdot (n_1, n_2)(p_1, p_2) = (tn_1p_1, tn_2p_2)$. The unipotent radical $N \times N$ has finitely many orbits $X_i^0 \times X_j^0$ on $X \times X$, with closures $X_i \times X_j$, so $H_*^T(X \times X)$ is a free H_T^* -module with basis $[X_i \times X_j]_T$. By Proposition 2.2, if $x_i x_j = \sum_k a_{ij}^k x_k$ then $\delta_*[X_k]_T = [\delta(X_k)]_T =$

$\sum_{ij} a_{ij}^k [X_i \times X_j]_T$. The coefficients a_{ij}^k are uniquely determined by the expansion of $\delta_*[X_k]_T$ because the classes $[X_i \times X_j]$ are linearly independent over H_T^* . By Theorem 3.2, these coefficients can be written as monomials in $\alpha_1, \dots, \alpha_d$ with nonnegative integer coefficients, where $\alpha_1, \dots, \alpha_d$ are the weights of T on $\text{Lie}(N \times N)$ (which are the same as the weights of T on \mathfrak{n}). \square

Proof of Theorem 3.2: First consider the case where $\dim N = 1$; then $B/T \xrightarrow{\sim} N \xrightarrow{\varphi} \mathbb{G}_a$, where $\mathbb{G}_a \cong \mathbb{A}^1$ is the additive group. Write $\alpha = \alpha_1$. We have $B = NT$, and the map $B/T \xrightarrow{\sim} N$ sends $nT \rightarrow n$. Now, B acts on B/T by left multiplication. Via the isomorphism of B/T with N , we obtain an action of B on N ; the subgroup $T \subset B$ acts on N by conjugation, and the subgroup N acts by left multiplication. The action of T by conjugation on N corresponds under φ to an action of T on \mathbb{A}^1 with weight α . Embed $B/T \hookrightarrow \mathbb{P}^1$ by $nT \mapsto [\varphi(n) : 1]$. The action of B on B/T extends to an action on \mathbb{P}^1 : the element $tn \in B$ acts by the matrix

$$\begin{pmatrix} \alpha(t) & \varphi(n) \\ 0 & 1 \end{pmatrix}.$$

The point $\infty = [1 : 0]$ is fixed by B , while the point $0 = [0 : 1]$ is fixed by T .

Now, B acts on $B \times^T X$ by left multiplication: $b \cdot (b', x) = (bb', x)$. Under the isomorphism $\theta : B \times^T X \rightarrow B/T \times X$ taking (b, x) to (bT, bx) , the B -action corresponds to the product action on $B/T \times X$. This extends to a B -action on $\mathbb{P}^1 \times X$. The projections $\pi : \mathbb{P}^1 \times X \rightarrow \mathbb{P}^1$ and $\rho : \mathbb{P}^1 \times X \rightarrow X$ are B -equivariant.

If $Y \subset X$ is a T -invariant subvariety then $B \times^T Y$ is a B -invariant subvariety of $B \times^T X$. Let Z be the Zariski closure of $\theta(B \times^T Y)$ in $\mathbb{P}^1 \times X$; $\theta(B \times^T Y)$ and Z are B -invariant subvarieties of $\mathbb{P}^1 \times X$. Let π_Z denote the restriction of π to Z .

Let $[w_0 : w_1]$ be projective coordinates on \mathbb{P}^1 , and w the rational function $\frac{w_0}{w_1}$. Let $g = \pi_Z^* w$; then w (and hence g) are rational functions which are T -eigenvectors of weight $-\alpha$. By [Br, Theorem 2.1]² we have in $H_*^T(\mathbb{P}^1 \times X)$ the relation $[\text{div}_Z g]_T = \alpha[Z]_T$. Therefore, in $H_*^T X$ we have the relation

$$\rho_*[\text{div}_Z g]_T = \alpha \rho_*[Z]_T \quad (3.4)$$

Now, $\pi_Z^{-1}(0) = \{0\} \times Y$ (cf. [KN]). Also, $\pi_Z^{-1}(\infty) = \{\infty\} \times D$ where D is a subscheme of X . Therefore (3.4) yields

$$[Y]_T = [D]_T + \alpha \rho_*[Z]_T$$

As π_Z is B -equivariant, and $\infty \in \mathbb{P}^1$ is B -fixed, it follows that $\{\infty\} \times D$, and hence D , are B -invariant. Each irreducible component D_i ($i = 1, \dots, r$) of D is therefore B -invariant (as B is connected) and if m_i is the multiplicity of D_i in D then $[D]_T = \sum_{i=1}^r m_i [D_i]_T$. Likewise, ρ is B -equivariant and Z is B -invariant. If Z_i is a component of Z then the map $\rho|_{Z_i}$ of Z_i onto its image in X is finite if and only if the map $\rho_T|_{Z_{iT}}$ of Z_{iT} onto its image in X_T is finite, and the degrees of the maps are the same. If we list the components of $\rho(Z)$ which are finite images of components of Z as D_{r+1}, \dots, D_s , it follows that each of

²Brion is using the convention that if X is a T -space, then T acts on functions on X by $(t \cdot f)(x) = f(tx)$, while we are using the convention that T acts on functions by $(t \cdot f)(x) = f(t^{-1}x)$. Under Brion's convention, our function g would be an eigenvector of weight α .

these components is B -invariant and that $\rho_*[Z]_T = \sum_{i=r+1}^s m_i[D_i]_T$ where m_i are positive integers. We conclude that

$$[Y]_T = \sum_{i=1}^r m_i[D_i]_T + \sum_{i=r+1}^s m_j \alpha[D_i]_T \quad (3.5)$$

where the D_i are B -invariant. This proves the result if $\dim N = 1$.

To prove the result in general, we can find a subgroup $N' \subset N$ such that N' is normal in B and $\dim N/N' = 1$. Let α be the weight of T on $\text{Lie}(N/N')$. Define $B' = N'T \subset B = NT$. By induction, we may assume the result is true for B' . It is enough to show that given a B' -invariant subvariety $Y \subset X$, we can write $[Y]_T$ as in (3.5), with B -invariant D_i . For this we modify the above proof, as follows. Replace B/T , $B \times^T X$, and $B \times^T Y$ by B/B' , $B \times^{B'} X$, and $B \times^{B'} Y$; the map θ now takes $B \times^{B'} X$ to $B/B' \times X$. Again $\varphi : B/B' \xrightarrow{\cong} \mathbb{G}_a = \mathbb{A}^1$ and T acts by weight α on \mathbb{A}^1 . We can embed $B/B' \hookrightarrow \mathbb{P}^1$ as before; the point $\infty = [1 : 0]$ is fixed by B , and $[0 : 1]$ is fixed by B' . With these modifications, (3.5) is proved as above. This proves the theorem. \square

4 Schubert varieties

4.1 Peterson's conjecture

Let G be a complex semisimple group and $B \supset T$ a Borel subgroup and maximal torus, respectively. Let N be the unipotent radical of B ; let $B^- = TN^-$ be the opposite Borel. Choose a system of positive roots so that the roots in \mathfrak{n} are positive. Let $W = N(T)/T$ denote the Weyl group; we abuse notation and write w for an element of W and also for a representative in $N(T)$. Let $X = G/B$ the flag variety. The T -fixed points are $\{wB\}_{w \in W}$; let $X_w^0 = N \cdot wB \subset X$ and $Y_w^0 = N^- \cdot wB$. Then $X = \coprod_w X_w^0$ (resp. $X = \coprod_w Y_w^0$) is a decomposition of X as a disjoint union of finitely many N (resp. N^-)-orbits. Let X_w and Y_w denote the closures of X_w^0 and Y_w^0 , and $\{x_w\}$ and $\{y_w\}$ the bases of $H_T^* X$ dual (in the sense of Proposition 2.1) to $\{[X_w]_T\}$ and $\{[Y_w]_T\}$.

Let $\alpha_1, \dots, \alpha_\ell$ denote the simple roots. Any weight of T on \mathfrak{n} (resp. \mathfrak{n}^-) is a nonnegative (resp. nonpositive) linear combination of the simple roots. Therefore, the next corollary is an immediate consequence of Theorem 3.1.

Corollary 4.1 *With notation as above, write $x_u x_v = \sum_w a_{uv}^w x_w$ and $y_u y_v = \sum_v b_{uv}^w y_w$, with a_{uv}^w and b_{uv}^w in H_T^* . Then a_{uv}^w (resp. b_{uv}^w) is a linear combination of monomials in the α_i , with nonnegative (resp. nonpositive) coefficients. \square*

Remark. Theorem 3.1 can be applied to the varieties X_w and Y_w , which are in general singular, to yield an analogue of Corollary 4.1 for $H_T^*(X_w)$ and $H_T^*(Y_w)$. The analogous result also holds for partial flag varieties.

Because X is smooth, the map $H_T^*(X) \xrightarrow{\cap[X]_T} H_*^T(X)$ is an isomorphism. The next lemma is known (cf. [P]) but for lack of reference we give a proof.

Lemma 4.2 *The map $H_T^*(X) \xrightarrow{\cap[X]_T} H_*^T(X)$ takes y_w to $[X_w]_T$.*

Proof: We can identify $H_T^*(X)$ with $\text{Hom}_{H_T^*}(H_T^T(X), H_T^*)$ (see the proof of Proposition 2). Hence, any $\gamma \in H_T^*(X)$ is uniquely determined by the values $\pi_*^T(\gamma \cap h')$ as h' ranges over the basis $\{[Y_{w'}]_T\}$ of $H_T^T(X)$.

Now, if $\gamma \in H_T^*(X)$ satisfies $\gamma \cap [X]_T = h$, then $\gamma \cap h' = h \cdot h'$. Indeed, the intersection product on $H_T^T(X)$ satisfies: if $\gamma' \cap [X]_T = h'$, then $\gamma \cdot \gamma' \cap [X]_T = h \cdot h'$; but $\gamma \cdot \gamma' \cap [X]_T = \gamma \cap (\gamma' \cap [X]_T) = \gamma \cap h'$.

Combining these facts, we see that to show $y_w \cap [X]_T = [X_w]_T$, it suffices to show

$$\pi_*^X([X_w]_T[Y_{w'}]_T) = \pi_*^X(y_w \cap [Y_{w'}]_T) = \delta_{ww'}.$$

Now, for any w, w' , the intersection $X_w \cap Y_{w'}$ is T -invariant, and is known to satisfy $\text{codim } X_w \cap Y_{w'} = \text{codim } X_w + \text{codim } Y_{w'}$. (Indeed, by [KL], $X_w \cap Y_{w'}^0$ is irreducible and of dimension $\dim X - \dim X_w - \dim Y_{w'}$, but by [F, p. 137], each component of $X_w \cap Y_{w'}$ has at least that dimension. It follows that $X_w \cap Y_{w'}^0$ is dense in $X_w \cap Y_{w'}$.) Hence $[X_w]_T[Y_{w'}]_T$ is a multiple of $[X_w \cap Y_{w'}]_T$. If $\dim X_w \cap Y_{w'} > 0$, then $\dim(X_w \cap Y_{w'})_T > \dim BT$, so $\pi_*^X([X_w \cap Y_{w'}]_T) = 0$. If $\dim X_w \cap Y_{w'} = 0$, then $w = w'$ and X_w and Y_w intersect with multiplicity 1 at the point wB [C, Prop. 2]. Hence $\pi_T^X : X_T \rightarrow BT$ maps $(X_w \cap Y_w)_T$ isomorphically onto BT , and therefore $\pi_*^X([X_w]_T[Y_w]_T) = \pi_*^X([X_w \cap Y_w]_T) = 1$. This proves the lemma. \square

The intersection product on $H_T^T(X)$ is induced by the product on $H_T^*(X)$, via the isomorphism $\cap[X]_T$. The above lemma and Corollary 4.1 therefore imply:

Corollary 4.3 *The intersection product on $H_T^T(X)$ is given by $[Y_u]_T[Y_v]_T = \sum_w a_{uv}^w [Y_w]_T$ (resp. $[X_u]_T[X_v]_T = \sum_w b_{uv}^w [X_w]_T$), where each a_{uv}^w (resp. b_{uv}^w) in H_T^* is a sum of monomials in the $\alpha_1, \dots, \alpha_\ell$, with nonnegative (resp. nonpositive) coefficients. \square*

Corollaries 4.1 and 4.3 were conjectured by Dale Peterson.

Example. As a concrete example, we work out the case of the flag variety of SL_2 . Here, we take B (resp. B^-, T) to be the upper triangular (resp. lower triangular, diagonal) matrices; we identify X with \mathbb{P}^1 , acting as usual. Then $W = \{1, s\}$ and the Schubert varieties are $X_1 = [1 : 0]$, $X_s = X$, $Y_1 = X$, $Y_s = [0 : 1]$. The character group of T is $\hat{T} = \mathbb{Z} \cdot x \cong \mathbb{Z}$, and the positive root is $\alpha = 2x$. The ring $H_T^* = \mathbb{C}[x]$. The action of T on \mathbb{P}^1 is with weights ± 1 , so $H_T^*X = \mathbb{C}[x, h]/(h+x)(h-x)$. We will identify H_T^*X with $H_T^T X$ via $\cap[X]_T$. Under this isomorphism, $[X_s]_T = [Y_1]_T = 1$. If $[z_0 : z_1]$ are projective coordinates on \mathbb{P}^1 , then z_0 may be viewed as a section of $\mathcal{O}(1)$ which is a T -eigenvector of weight -1 . Then $z_0 \otimes 1$ is a T -invariant section of $\mathcal{O}(1) \otimes \mathbb{C}_1$ (here \mathbb{C}_1 is the trivial line bundle with T with weight 1). The zero-scheme of $z_0 \otimes 1$ is $[0 : 1]$, so we conclude $[Y_s]_T = [0 : 1]_T = c_1^T(\mathcal{O}(1) \otimes \mathbb{C}_1) = h + x$. Similarly, $[X_1]_T = h - x$. The only interesting multiplication among the classes $[X_w]_T$ is

$$[X_1]_T \cdot [X_1]_T = (h - x)^2 = h^2 - 2hx + x^2 = 2x^2 - 2hx = -2x(h - x) = -\alpha[X_1]_T.$$

Similarly, the only interesting multiplication among the classes $[Y_w]_T$ is

$$[Y_s]_T[Y_s]_T = \alpha[Y_s]_T.$$

These agree with Corollaries 4.1 and 4.3.

4.2 Billey's conjecture

Kostant and Kumar [KK] defined functions (for each $w \in W$) $\xi^w : W \rightarrow S(\hat{T}) \subset S(\mathfrak{t}^*)$, and showed that for any $u, v \in W$, one can write

$$\xi^u \xi^v = \sum_w p_w^{uv} \xi^w$$

for unique $p_w^{uv} \in S(\mathfrak{t}^*)$. Billey [Bi] observed in examples that if $\nu \in \mathfrak{t}$ satisfies $\alpha(\nu) > 0$ for all positive roots α , then $p_w^{uv}(\nu) \geq 0$, and asked if a geometric proof was possible.

Arabia [A] proved the following relation of the functions ξ^w to the T -equivariant equivariant cohomology of the flag variety. We use the notation of the preceding subsection: thus, $i_w : wB \rightarrow G/B = X$ denotes the inclusion, and $i_w^* : H_T^*(X) \rightarrow H_T^*(wB) = H_T^*$ the pullback. As usual, we identify $H_T^*(X)$ with $H_*^T(X)$.

Theorem 4.4 (1) $i_u^* x_w = \xi^{w^{-1}}(u^{-1})$.

$$(2) p_{w^{-1}}^{u^{-1}, v^{-1}} = a_{uv}^w.$$

This is proved (in the general Kač-Moody case) in [A, Theorem 4.2.1]. We have stated this theorem using the conventions of [KK] for the functions ξ^w ; below we explain the relationship between the conventions of [A] and [KK]. Note that (2) follows immediately from (1), since (as noted by Arabia) the pullback $\oplus i_w^* : H_T^*(X) \rightarrow \oplus H_T^*$ is injective.

As a consequence, we obtain Billey's conjecture:

Corollary 4.5 *If $\nu \in \mathfrak{t}$ satisfies $\alpha(\nu) > 0$ for all positive roots α , then $p_w^{uv}(\nu) \geq 0$.*

Proof: This follows immediately from the preceding corollary and Corollary 4.3. \square

We now discuss the conventions of [A] and [KK]. Let $\mathbb{C}[W]$ denote the group algebra over \mathbb{C} of W ; let Q be the quotient field of $S(\mathfrak{t}^*)$. Kostant and Kumar set $Q_W = \mathbb{C}[W] \otimes Q$; Arabia defines Q and Q_W with rational rather than complex coefficients, but we will ignore this difference. Both [KK] and [A] define elements $\xi^w \in \text{Hom}_Q(Q_W, Q)$, but with different conventions: if we use ξ^w for the elements defined in [KK] and ξ_A^w for the elements defined in [A], then $\xi^w = \xi_A^{w^{-1}}$.

Let $F(W, Q)$ denote the set of functions from W to Q . Both [KK] and [A] use identifications $F(W, Q) \xrightarrow{\sim} \text{Hom}_Q(Q_W, Q)$; we will denote their respective identifications by

$$\begin{aligned} f &\mapsto f_K, & \text{where } f_K(\delta_u \otimes 1) &= f(u) & [\text{KK}, (4.17)] \\ f &\mapsto f_A, & \text{where } f_A(\delta_u \otimes 1) &= f(u^{-1}) & [\text{A}, \text{Section 4.1}]. \end{aligned}$$

If we define f^w and g^w in $F(W, Q)$ by $f_K^w = \xi^w$, $g_A^w = \xi_A^w$, then $f^w(u) = g^{w^{-1}}(u^{-1})$.

Arabia uses the injection

$$\oplus i_u^* : H_T^*(X) \hookrightarrow \oplus H_T^* \simeq F(W, S(\mathfrak{t}^*)) \subset F(W, Q)$$

to identify $H_T^*(X)$ with a subset of $F(W, Q)$. In his paper, he proves that under this identification, g^w corresponds to what we have denoted by $x_w \in H_T^*(X)$. In [KK] there is no separate notation introduced for the f^w , but rather they are identified with ξ^w , i.e., the ξ^w are viewed as elements of $F(W, Q)$. If we return to their notation, we see $\xi^{w^{-1}}(u^{-1}) = i_u^* x_w$, as stated in Theorem 4.4.

Note that if we let ξ_B^w denote the functions used by Billey, then $\xi_B^w(u) = \xi^{w^{-1}}(u^{-1})$.

4.3 The Kač-Moody case

The analogues of Corollaries 4.1 and 4.5 are also valid for flag varieties (complete or partial) of Kač-Moody groups. The key point is that such a flag variety, although in general infinite dimensional, can be approximated by finite dimensional varieties for which the hypotheses of Theorem 3.1 are satisfied. Indeed, this was exactly the geometric motivation of Kumar and Nori. We will briefly sketch how this works in equivariant cohomology. The basic facts we need can be found in [Sl], to which we refer for a more detailed explanation of the notation. Let G be a Kač-Moody group and B a Borel subgroup; let $X = G/B$ denote the flag variety. The group B is a proalgebraic group (inverse limit of algebraic groups), and it has a Levi decomposition $B = TN$, where N is a proalgebraic pronipotent group (denoted by U in [Sl] and [KN]) and T is a finite dimensional torus. The space X has the structure of ind-variety: it is realized as a union $X = \cup_{k \geq 0} X_k$, where each X_k is a finite dimensional variety embedded as a closed subvariety of X_{k+1} . Here X_k is defined as follows. We have $X = \coprod X_w^0$, realizing X as a disjoint union of Schubert cells $X_w^0 = B \cdot wB$. The union is over all elements of the Weyl group W ; each X_w^0 is isomorphic to the affine space $\mathbb{A}^{l(w)}$, where $l(w)$ is the length of w . By definition, $X_k = \coprod_{l(w) \leq k} X_w^0$; this is a finite dimensional projective variety which is paved by affines. Moreover, each X_k is B -stable, and there exists a subgroup $N_k \subset N$, normal in B , such that $B_k = B/N_k$ is a finite dimensional solvable group, and the action of B on X_k factors through the map $B \rightarrow B_k$. Each X_k therefore satisfies the hypotheses of Theorem 3.1. As in the finite case, there is a set of simple roots $\alpha_1, \dots, \alpha_l$ in \mathfrak{t}^* , and moreover, for any k , every weight in $\text{Lie}(N/N_k)$ is a nonnegative linear combination of simple roots.

Now, for any fixed i , the pullback $H_T^i(X) \rightarrow H_T^i(X_k)$ is a canonical isomorphism for k sufficiently large (as the decomposition of X into Schubert cells makes X a CW-complex, and X_k contains all cells in X of dimension $\leq 2k$, and similarly for the mixed spaces X_{kT} and X_T). There is a basis $\{x_w\}$ of $H_T^*(X)$ dual to the fundamental classes $[X_w]_T$, in the sense that the pullbacks to $H_T^*(X_k)$ form a basis dual to the $[X_w]_T \in H_*^T(X_k)$, for $l(w) \leq k$. This basis does not depend on k , as can be seen using property (2.2) of the pairing, applied to the inclusion map of X_k into X_{k+1} . Theorem 3.1 therefore implies the following corollary, also conjectured by Peterson.

Corollary 4.6 *With notation as above, if X is the flag variety of a Kač-Moody group, with basis $\{x_w\}$ of $H_T^*(X)$, then $x_u x_v = \sum_w a_{uv}^w x_w$, with $a_{uv}^w \in H_T^*$ a linear combination of monomials in the α_i , with nonnegative coefficients. \square*

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